Modulated phases of a one-dimensional sharp interface model in a magnetic field

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We investigate the ground states of one-dimensional continuum models having short-range ferromagnetic-type interactions and a wide class of competing longer-range antiferromagnetic-type interactions. The model is defined in terms of an energy functional, which can be thought of as the Hamiltonian of a coarse-grained microscopic system or as a mesoscopic free-energy functional describing various materials. We prove that the ground state is simple periodic whatever the prescribed total magnetization might be. Previous studies of this model of frustrated systems assumed this simple periodicity but, as in many examples in condensed-matter physics, it is neither obvious nor always true that ground states do not have a more complicated, or even chaotic structure.

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I. INTRODUCTION

In two previous papers[1,2] we considered one-dimensional (1D) discrete and continuum models of classical spin systems with short- and long-range competing interactions. We proved that, if the long-range interactions are reflection positive and the short-range interaction is ultralocal [nearest neighbor (nn) in the lattice case] then the ground states of the system display periodic striped order. The proof was based on antiferromagnetic reflections about the nodes of the spin-density configuration, and used the fact that no external magnetic field was imposed or, equivalently, that the total magnetization was zero. In this note, we extend the analysis of Refs. 1 and 2 to a continuum sharp interface model in the case of nonzero magnetization. We find that for a large class of antiferromagnetic long-range interactions with arbitrarily fixed total magnetization, all the ground states are *simple periodic*, i.e., they consist of a sequence of blocks of alternate sign of the spin and alternate lengths $\ldots,\ell_1,\ell_2,\ell_1,\ell_2,\ldots,$ so that the magnetization per unit length, which is specified, is $m=(\ell_1-\ell_2)/(\ell_1+\ell_2)$. Recently, Nielsen *et al.*[3] studied the dependence of the period $\ell_1+\ell_2$ on the surface tension in such a 1D sharp interface model with power-law interactions, under the assumption (supported by numerical evidence) that all the ground states of the system are simple periodic. One of our goals here is to prove that their restriction to simple periodicity is justified.

If we give up the continuum nature of the model then, in general, the simple periodic states are not expected to be the ground states of the system. Indeed, for a discrete Ising model with only long-ranged antiferromagnetic convex interactions, the ground states display a complex structure as a function of the prescribed magnetization. See Refs. 4–6 If, in this discrete model, a short-range ferromagnetic interaction of strength J is added to the long-range antiferromagnetic one, it is reasonable to expect that the ground states still display a similar (periodic or quasiperiodic) complex structure. As J becomes large, the typical scale ℓ of modulation increases (in particular, for J sufficiently large, ℓ is much larger than the lattice spacing), and the ground states of the discrete model are close to those of a continuum model with sharp interfaces, modulo small displacements of the locations of the domain walls. It is *a priori* unclear whether the ground states of the continuum model should display the same complex structures expected in the discrete case or not.

Simple periodicity cannot, therefore, be taken for granted, and since the numerical tests commonly investigate only the local stability of not-too-complex periodic structures, it is desirable to have a rigorous proof of simple periodicity. In this paper we provide such a proof for reflection positive potentials (including the power-law potentials considered in Ref. 3) and for perturbations of reflection positive potentials. Indeed the number of physical models for which periodicity can be rigorously proved is very small,[7–9] and our methods here might lead to other useful examples. This is of particular interest in 2D, where mesoscopic free-energy functionals of the type we consider here have been proposed as models for micromagnets,[10–12] diblock copolymers,[13–15] anisotropic electron gases,[16,17] polyelectrolytes,[18] charge-density waves in layered transition metals[19] and superconducting films.[20] In all these systems, existence of simple periodic ground states has been argued heuristically,[10,12–14,16,17,20–23] but there are at present only few rigorous results.[11,24–27] Note, however, that our results on the structure of the one-dimensional ground states also apply to higher dimensional cases, if we restrict to one-dimensional configurations, i.e., to configurations that are translational invariant in $d-1$ coordinate directions. In other words, the results of this paper imply that in a continuum model with sharp interfaces and long-range reflection positive interactions (or perturbations of reflection positive interactions) in dimension $d\geq 1$, the minimal energy state among the one-dimensional configurations displays simple periodicity.

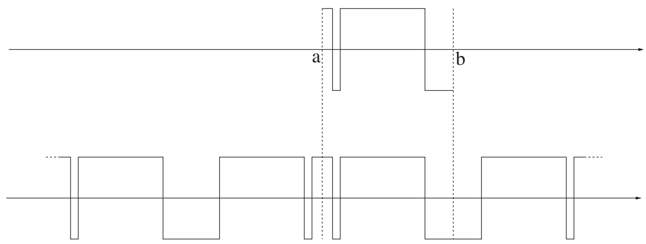

FIG. 1. A function defined in the interval $[a,b]$ (upper part) and its Neumann extension (lower part).

The paper is organized as follows. In Sec. II we define the model, state the main results in the form of two theorems, and discuss their significance. In Sec. III we prove the first theorem, for the case of reflection positive interactions. The proof combines ideas from our previous papers and from Refs. 7–9. In Sec. IV we prove stability of our results, namely, that small perturbations of reflection positive interactions do not affect the simple periodicity of the ground state; moreover, we discuss the ground-state properties of the system at small J. In Sec. VI we summarize our results and discuss conclusions and perspectives. In appendix we prove some nondegeneracy properties of the minimizers, used in the proof of Theorem 1.

II. MAIN RESULTS

Given $L>0$, we consider the following energy functional:

$$\mathcal{E}(u)=\frac{J}{2}\int_0^L dx|u'|+\frac{1}{2}\int_0^L dx\int_{-\infty}^{+\infty}dy u(x)v(x-y)\tilde{u}(y), \tag{2.1}$$

where $J>0$, v is a positive potential, and u is a function defined for $0\le x\le L$ that assumes the values ± 1, representing the configurations of our 1D magnetic system, and u' is its derivative. For any function u with values ± 1, $\int_0^L dx|u'|$ is simply twice the number of times $u(x)$ jumps from $+1$ to -1 or from -1 to $+1$.

The function $\tilde{u}$, in Eq. (2.1), is the (Neumann) extension of u over the whole real axis and is defined as follows. Given a function w defined in an interval $I=[a,b]$, its Neumann extension $\tilde{w}$ is obtained from w by iteratively reflecting it about the end points a and b of I and about their images, see Fig. 1.

We will also assume that u satisfies the magnetization constraint

$$\frac{1}{L}\int_0^L dx u(x)=m,\quad 0\le m<1. \tag{2.2}$$

In the following, we shall require that the potential v satisfies some positivity properties. More precisely, we shall consider

(1) Reflection positive potentials, i.e.,

$$v(x)=\int_0^\infty d\alpha\mu(\alpha)e^{-\alpha|x|}, \tag{2.3}$$

with μ a positive density such that v is integrable, i.e., $\int_0^\infty d\alpha\mu(\alpha)\alpha^{-1}<\infty$; Eq. (2.3) is equivalent to the condition that v is completely monotone, i.e., $(-1)^n\frac{d^n v(x)}{dx^n}\ge 0$, for all $x>0$, $n\ge 0$;[28]

(2) Finite-range perturbations of reflection positive potentials, i.e.,

$$v(x)=v_0(x)+f_\varepsilon(x), \tag{2.4}$$

with v_0 as in Eq. (2.3) and f_ε a finite, even potential, with range ε.

Our first result is that in the case of reflection positive interactions the minimizers of Eq. (2.1) are simple periodic, for all $J>0$.

Theorem 1 [Simple periodicity]. Given an integer M and $0\le m<1$, let $u_{M,m}(x)$ be defined for $0\le x\le L/M$ by

$$u_{M,m}(x)=\begin{cases}+1 & \text{if}\quad 0\le x\le \frac{1+m}{2}\frac{L}{M},\\ -1 & \text{if}\quad \frac{1+m}{2}\frac{L}{M}\le x\le \frac{L}{M}.\end{cases} \tag{2.5}$$

Then all the finite volume minimizers of Eq. (2.1), with reflection positive potential Eq. (2.3), are of the form $w_M^+(x)=\tilde{u}_{M,m}(x)$ or $w_M^-(x)=\tilde{u}_{M,m}(x-\frac{L}{M})$, with M fixed by the variational equation

$$\mathcal{E}(w_M^\pm)=\min_{M'}\mathcal{E}(w_{M'}^\pm), \tag{2.6}$$

where M' is a positive integer.

The variational Eq. (2.6) has been studied and solved, for some explicit choices of v, in Ref. 3. One might worry about the fact that the resulting picture of a zero-temperature phase diagram consisting of simple periodic ground states crucially depends on the choice of a reflection positive, or at least convex, potential. Any reflection positive potential is convex and any convex potential that goes to zero at infinity has a cusp at $x=0$. A natural question, therefore, is whether the cusp plays an important role or not in the resulting phenomenon. It is reassuring that we can prove that the simple periodicity property is stable under small perturbations f_ε of the reflection positive potential that remove the cusp, as long as ε is smaller than the resulting period.

Theorem 2 [Perturbative stability]. Let v, v_0, and f_ε be defined as in Eq. (2.4) and let us assume that

$$\varepsilon<\frac{J}{\int_{-\infty}^{\infty}dx[v_0(x)+2|f_\varepsilon(x)|]}. \tag{2.7}$$

Then the finite volume minimizers of Eq. (2.1) with perturbed reflection positive potential (2.4) are functions of the form $w_M^\pm$, with $w_M^\pm$ defined as in Theorem 1, and with M fixed by the variational equation

$$\mathcal{E}(w_M^\pm)=\min_{M'}\mathcal{E}(w_{M'}^\pm). \tag{2.8}$$

Theorem 2 can be interpreted as saying that for any finite J the simple periodicity property is stable under small finite-range perturbations of the potential. It can also be interpreted the other way round: For any given finite-range perturbation of a reflection positive potential, the ground state is simply periodic if J is large enough. In this sense, it suffices that

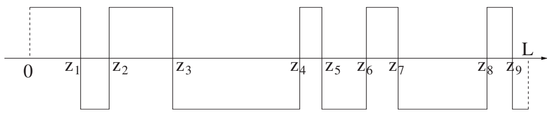

FIG. 2. A putative minimizer w of $E_\alpha(u)$ in the subspace of functions with $M=9$ jumps, and its sequence of nondegenerate jump points.

beyond a certain range, which grows with J, the interaction is reflection positive, in order for the ground state to be simply periodic. On the contrary, at small values of J, the structure of the ground states may depend critically on the short range properties of the potential, as discussed in Sec. IV, after the proof of Theorem 2.

A similar stability result is valid for *lattice models* in zero magnetic field with zero (or unconstrained) magnetization. Consider a 1D Ising model with finite range ferromagnetic interactions and long-range antiferromagnetic reflection positive interactions. If the strength J of the nn ferromagnetic interaction is large enough, while the strength of further neighbor ferromagnetic and long-range antiferromagnetic interactions are kept fixed, the ground states are simple periodic. This extends the results of Ref. 1, where simple periodicity was proved only for the case of nn ferromagnetic interactions. The proof of this claim goes along the same lines as the proof of Theorem 2 and we will not belabor its details here. On the contrary, our proof does not extend to the lattice case if the prescribed magnetization is different from zero. We do not think that this is just a technical problem: as remarked in the introduction, the ground states of discrete models with prescribed magnetization different from zero are not expected to be simple periodic, see Refs. 4–6, so we expect the analogs of Theorems 1 and 2 to be false in this case.

III. PROOF OF THEOREM 1

Let us first fix an integer M and let us temporarily restrict ourselves to functions with exactly M jumps in $[0,L]$. Let us rewrite the energy of such functions in the form

$$\mathcal{E}(u) = JM + \frac{1}{2}\int_0^\infty d\alpha\mu(\alpha)E_\alpha(u), \quad E_\alpha(u) = \int_0^L dxu(x)W_{\alpha,u}(x), \tag{3.1}$$

where

$$W_{\alpha,u}(x) = \int_{-\infty}^{+\infty} dye^{-\alpha|x-y|}\tilde{u}(y) \tag{3.2}$$

is the potential at point x associated to the exponential interaction $e^{-\alpha|x-y|}$. A short calculation shows that $W_{\alpha,u}$ satisfies the linear second-order equation

$$W''_{\alpha,u}(x) - \alpha^2 W_{\alpha,u}(x) = -2\alpha u(x). \tag{3.3}$$

For a given M and m exactly one simple periodic function exists (up to translations). We are going to prove that for each $\alpha>0$, $E_\alpha(u)$ is minimized by this simple periodic function and, therefore, $\mathcal{E}(u)$ is also minimized by this function.

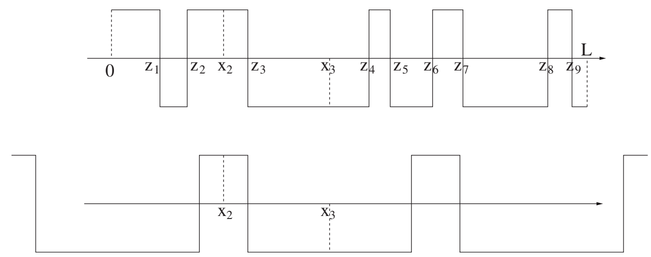

FIG. 3. A putative minimizer w with $M=9$ jumps (upper part). If x_2 and x_3 are zero derivative points of w, then the potential generated by w and by $\tilde{w}_3$ (lower part) inside the interval $[x_2,x_3]$ are the same.

Let us now fix α and let w be a minimizer of $E_\alpha(u)$ in the space of functions with exactly M jumps. We can assume, without loss of generality, that $w(0)=+1$. In this case, w is uniquely determined by the sequence of its jump points $0 \le z_1 \le z_2 \le \cdots \le z_M \le L$, see Fig. 2; these jump points have to satisfy a constraint induced by Eq. (2.2)

$$z_1 - (z_2 - z_1) + \cdots + (-1)^{M-1}(z_M - z_{M-1}) + (-1)^M(L - z_M) = Lm. \tag{3.4}$$

The existence of a minimizer for fixed α and fixed number of jumps is proved in appendix, where it is shown in particular that any such minimizer has a nondegenerate sequence of jump points, i.e., $0<z_1<z_2<\cdots<z_M<L$, and that the potential at the jump points is constant, i.e., $W_{\alpha,w}(z_i)$ is independent of i. As discussed in appendix, the potential $W_{\alpha,w}$ is strictly convex in the intervals where w is negative and concave in the intervals where w is positive. Therefore, $W_{\alpha,w}$ has exactly one zero derivative point in each interval (z_i,z_{i+1}), $i=1,\ldots,M-1$; let us denote it by x_i, $x_i\in(z_i,z_{i+1})$. We also define $x_0=0$ and $x_M=L$; note that, by the Neumann's boundary conditions imposed on the big box $[0,L]$, we also have that $W'_{\alpha,w}(x_0)=W'_{\alpha,w}(x_M)=0$ (Fig. 3).

The ordered (and nondegenerate) sequence of points x_i, $i=0,\ldots,M$, induces a partition of $[0,L]$ in intervals $I_i=[x_i,x_{i+1}]$ characterized by the fact that $W'_{\alpha,w}(x_i)=0$. Now, the first key remark, due to Müller and to Chen and Oshita,[7,9] is that, for every $x\in I_i$, $W_{\alpha,w}(x)=W_{\alpha,w_i}(x)$, where $w_i=\tilde{w}_{I_i}$, with w_{I_i} the restriction of w to I_i. In other words the claim is that, if we restrict to intervals whose end points are zero derivative of the potential, then the potential inside such an interval is the same as one would get by repeatedly reflecting w_{I_i} about the end points of I_i and about their images under reflections. The reason is very simple: both $W_{\alpha,w}(x)$ and $W_{\alpha,w_i}(x)$ satisfy the same Eq. (3.3) in the same interval, with $W'=0$ boundary conditions at x_i and x_{i+1}. The solution of the linear Eq. (3.3) with these boundary conditions is unique, which means that the two potentials must be the same on I_i. Therefore,

$$\int_0^L dxw(x)W_{\alpha,w}(x) = \sum_{i=0}^{M-1}\int_{x_i}^{x_{i+1}} dxw_i(x)W_{\alpha,w_i}(x). \tag{3.5}$$

On the other hand, denoting by p_i, q_i the lengths of the positive and negative parts of w_i on I_i, respectively, a computation shows that

$$\alpha^2\int_{x_i}^{x_{i+1}} dx w_i(x) W_{\alpha,w_i}(x) = 2p_i\alpha + 2q_i\alpha - 4\frac{\sinh(\alpha p_i)\sinh(\alpha q_i)}{\sinh(\alpha p_i+\alpha q_i)} \equiv f(\alpha p_i,\alpha q_i). \tag{3.6}$$

It is straightforward to check that f is a jointly strictly convex function of the variables (p,q), that is, the second derivative matrix (the Hessian) of $f(x,y)$, which is

$$H(f)(x,y) = \frac{8}{[\sinh(x+y)]^3} \times \begin{pmatrix} (\sinh y)^2\cosh(x+y) & -\sinh x\cdot\sinh y \\ -\sinh x\cdot \sinh y & (\sinh x)^2\cosh(x+y)\end{pmatrix}, \tag{3.7}$$

is positive definite for all $x,y>0$. The convexity implies that the minimum energy occurs when all the p_i and q_i are the same, given the constraint on their sums. Thus, the potential energy at fixed α of a minimizer ϕ in the subspace of functions with M jumps satisfies

$$\alpha^2\int_0^L dx w(x) W_{\alpha,w}(x) = \sum_{i=0}^{M-1} f(\alpha p_i,\alpha q_i) \geq M f\left(\alpha\frac{\Sigma_i p_i}{M},\alpha\frac{\Sigma_i q_i}{M}\right) = M f\left(\alpha\frac{L}{M}\frac{1+m}{2},\alpha\frac{L}{M}\frac{1-m}{2}\right). \tag{3.8}$$

In the last equality we used the mass constraint (2.2). Note that the inequality in Eq. (3.8) is strict unless the values of (p_i,q_i) are independent of i. Now, the rhs of Eq. (3.8) is nothing else but $\int_0^L dx\int_{\mathbb{R}} dy w_M^{\pm}(x)e^{-\alpha(x-y)}\tilde{w}_M^{\pm}(y)$, with $w_M^{\pm}$ defined as in Theorem 1. This shows that the only two minimizers of $E_\alpha(u)$ on the subspace of functions with M jumps are precisely the $w_M^{\pm}$ defined in Theorem 1. Quite remarkably, these minimizers are independent of α: this is the second key remark. Therefore, averaging over α and minimizing over M, we get Theorem 1. Q.E.D.

Let us conclude this section by a comment. The proof of Theorem 1 raises the question of whether there might be nonsimple periodic "metastable" states w in which the potential at the jump points, $W_w(z_i)=\int_0^\infty d\alpha\mu(\alpha)W_{\alpha,w}(z_i)$, are all equal. A computation along the same lines of the proof of Theorem 1 allows one to prove that such metastable states do not exist when $v(x)=Ce^{-\alpha_0|x|}$ but we do not know whether these are possible for more general reflection positive (or just convex) potentials.

IV. PROOF OF THEOREM 2

Let us fix $J>0$ and let us consider a minimizer w of Eq. (2.1). Let M^* be its number of jumps and let $h_0=2z_1$, $h_1=z_2-z_1,\ldots,h_{M^*-1}=z_{M^*}-z_{M^*-1}$, $h_{M^*}=2(L-z_{M^*})$ be the corresponding block sizes. An important remark is that, for any fixed $J>0$, under the assumptions of Theorem 2, there is an *a priori* lower bound on the block sizes in the ground state. In fact, since w is an energy minimizer, energy must not decrease if we change sign of w in (z_i,z_{i+1}), i.e., in the block of size h_i. If ΔE denotes the energy change corresponding to such sign change, we have

$$0 \leq \Delta E \leq -2J + 4\int_{z_i}^{z_{i+1}} dx\int_{z_{i+1}}^{\infty} dy[v_0(x-y)+|f_\varepsilon(x-y)|]. \tag{4.1}$$

Since f_ε has range ε, we see that the rhs of Eq. (4.1) is bounded above by $-2J+2h_i\int_{-\infty}^{+\infty}dx|v_0(x)|+4\varepsilon\int_{-\infty}^{+\infty}dx|f_\varepsilon(x)|$, which implies

$$h_i \geq \frac{J-2\varepsilon\int_{-\infty}^{+\infty}dx|f_\varepsilon(x)|}{\int_{-\infty}^{+\infty}dx|v_0(x)|} \equiv h^*, \tag{4.2}$$

with $h^*>0$, by the assumptions of Theorem 2. It then must be true that $\frac{1-|m|}{2}\frac{L}{M^*}\geq h^*$.

If, as assumed in Theorem 2, the range ε of the perturbation f_ε is strictly smaller than h^*, then the contribution to the ground-state energy coming from f_ε is essentially trivial and is given by

$$\frac{1}{2}\int_0^L dx\int_{\mathbb{R}} dy w(x) f_\varepsilon(x-y)\tilde{w}(y) = L\int_{\mathbb{R}} dx f_\varepsilon(x) - 2M\int_0^\varepsilon dy\int_{-\varepsilon}^0 dx f_\varepsilon(y-x). \tag{4.3}$$

Therefore, defining $J_0=\frac{1}{2}\int_0^\varepsilon dy\int_{-\varepsilon}^0 dx f_\varepsilon(y-x)$, we can write,

$$\mathcal{E}(w) = L\int f_\varepsilon + (J-J_0)M + \frac{1}{2}\int_0^\infty d\alpha\mu(\alpha) w(x) W_{\alpha,w}(x). \tag{4.4}$$

Proceeding as in Sec. III, and using the fact that $\frac{1-|m|}{2}\frac{L}{M^*}\geq h^*>\varepsilon$, we find that the rhs of Eq. (4.4) is bounded from below by $\mathcal{E}(w_{M^*}^{\pm})$, as desired. As in the proof in Sec. III, the bound below is strict, unless $w=w_{M^*}^{\pm}$. This concludes the proof of Theorem 2. Q.E.D.

V. THE ZERO TEMPERATURE PHASE DIAGRAM WITH PERTURBED REFLECTION POSITIVE INTERACTIONS

Fix a perturbation f_ε. By Theorem 2, we know that for large enough J, the ground states are simply periodic. It is natural to ask what happens for smaller values of J. We claim that in this case the nature of the ground state critically depends on the short range properties of the potential and, more precisely, it depends on whether v is of positive type [i.e., its Fourier transform $\hat{v}(k)\geq 0$] or not. Before we enter a discussion of this claim, let us remark that even if f_ε is arbitrarily small, with an arbitrarily small range, the resulting potential $v=v_0+f_\varepsilon$ can be of either type, depending on the specific properties of f_ε. For example, let $v_0(x)=e^{-|x|}$, g_ε a positive compactly supported even function of range ε and $A^{-1}=\int_{-\infty}^{\infty}dx\cosh x g_\varepsilon(x)$, then the potential w, given by the convolution of Av_0 and g_ε, $w=Av_0 * g_\varepsilon$, is continuous, equal to

$e^{-|x|}$ for $|x|>\varepsilon$ and equal to $e^{-|x|}+O(\varepsilon^2)$ if $|x|\le\varepsilon$. Moreover, its Fourier transform has the same sign as that of $\hat{g}_\varepsilon$, which might or might not be positive. For example, the triangle function $g_\varepsilon(x)=\max\{0,\varepsilon-|x|\}$ has $\hat{g}_\varepsilon\ge 0$, while the square function $g_\varepsilon(x)=\theta(\varepsilon-|x|)$ is not of positive type.

Let us now explain why we expect that the nature of the ground state at small J depends critically on the positivity of $\hat{v}$. To gain some intuition about the problem we first look at the case $J=0$ and temporarily replace the constraint $|u(x)|=1$ by the softer one $|u(x)|\le 1$. In this case, if v is of positive type, then the potential term $\int_0^L dx\int_{-\infty}^{+\infty}dy u(x)v(x-y)\tilde{u}(y)$ is happiest when u is constant, i.e., $u\equiv m$. When $\min\hat{v}(k)=\hat{v}(k^*)<0$ then the potential energy wants u to be modulated at the wavelength k^*, e.g., $u=m+(\text{const.})\cos(k^*x)$.[21,23] [We warn the reader that this k^* has nothing to do with the spontaneous modulation wavelength resulting from the competition between surface tension and potential energy in models with long-range interactions and a soft constraint on $|u|$ given by a double-well potential, e.g., in models of the form $(\theta/\lambda)\int|u'|^2+\lambda\int(u^2-1)^2+\int u\cdot(v*u)$, with λ small. Our results and in particular the present discussion do not apply to this latter case, see the concluding remarks section for more comments about this point.]

We return to the case of a hard constraint $|u(x)|=1$: if $J=0$, the functional $\mathcal{E}(u)$ is not minimized by any specific function u. Instead, one can find a sequence of highly oscillating functions u_i that take only the values ± 1, approximating better and better as $i\to\infty$ the smooth functions $u\equiv m$ or $u=m+(\text{const.})\cos(k^*x)$, which make the energy $\mathcal{E}(u_i)$ closer and closer to its infimum, in the two cases where v is of positive type or not, respectively. In the presence of a small positive J, the functional $\mathcal{E}(u)$ admits nontrivial minimizers: they will be close to one of these highly oscillating configurations, with a finite (but possibly very small) oscillation scale. Therefore, if $|u|=1$ and v is not of positive type, the minimizer at small J will be close to a highly oscillating approximation of the modulated minimizer $m+(\text{const.})\cos(k^*x)$, and so *it will not be simply periodic*. If $|u|=1$ and v is of positive type, the minimizer at small J will be close to a highly oscillating approximation of the constant configuration $u\equiv m$, and it may very well be that the optimal u is simply periodic. We actually conjecture that this is the case.

To summarize: in the presence of a fixed finite range perturbation of a reflection positive interaction, we expect the zero-temperature phase diagram of the system to present qualitative differences depending on whether the resulting long-range interaction is of positive type or not. If v is of positive type, we expect that the ground state is simple periodic, for any $J>0$; if v is not of positive type, we expect the ground state to have transition from a nonsimple periodic state (with period essentially independent of J) to a simple periodic state.

VI. CONCLUSIONS

We investigated a one-dimensional continuum sharp interface model with long-range antiferromagnetic interactions, under the constraint that the total magnetization is prescribed, and in general different from zero. If the long-range interaction v is reflection positive, we proved that the ground states are simple periodic, i.e., they consist of a sequence of blocks of alternate sign of the spin and alternate lengths $\ldots,\ell_1,\ell_2,\ell_1,\ell_2,\ldots$; we also proved that simple periodicity is stable under small finite range perturbations of the reflection positive interaction. These results generalize previous results[1,2] concerning simple periodicity of the 1D ground states on the lattice or on the continuum with zero total magnetization. Moreover, they provide a rigorous justification that the ansatz chosen by Nielsen *et al.*[3] to investigate the dependence of the modulation length on the surface tension and on the decay rate of the long-range interaction is correct.

Our results imply analogous statements for the case of higher dimension $d\ge 1$, but only *if one restricts to 1D configurations*, i.e., to configurations that are translational invariant in $d-1$ dimensions. The conjecture that the ground states of $\mathcal{E}(u)$ in two or more dimensions are simple periodic, and possibly one-dimensional, is still open.

The fact that simple periodicity is stable under small finite range perturbations, naturally leads us to ask whether simple periodicity should also be expected at positive (possibly small) temperatures. Unfortunately, we are still unable to treat positive temperatures. At a mean field level, the effect of the temperature can be mimicked by a soft interface model, replacing Eq. (2.1), described by the functional[2]

$$\mathcal{F}_\lambda(u)=\frac{3J^2}{8\lambda}\int_0^L dx|u'|^2+\lambda\int_0^L dx(u^2-1)^2+\frac{1}{2}\int_0^L dx\int_{-\infty}^{+\infty}dy u(x)v(x-y)\tilde{u}(y), \qquad (5.1)$$

under the constraint $\int_0^L dx u(x)=mL$. $\mathcal{F}_\lambda(u)$ reduces to $\mathcal{E}(u)$ in the limit $\lambda\to\infty$, which corresponds to "zero temperature;" on the contrary, λ small roughly corresponds to the case of high temperature, in which case the scale of the modulation is dictated by the competition between the kinetic energy and the potential energy.[12] While the ground states of Eq. (5.1) with $m=0$ are simple periodic for any $\lambda>0$,[2] we are still unable to prove the same at $m\ne 0$ for a generic reflection positive interaction. Technically, the main difficulty in trying to generalize the proof of Theorem 1 to this case is that if, as in Sec. III, we rewrite the potential energy as a superposition of the potentials $W_{\alpha,u}(x)$ generated by the exponential interactions $\exp\{-\alpha(x-y)\}$, and we try to minimize the functional

$$\frac{3J^2}{8\lambda}\int_0^L dx|u'|^2+\lambda\int_0^L dx(u^2-1)^2+\frac{1}{2v(0)}\int_0^L u(x)W_{\alpha,u}(x), \qquad (5.2)$$

for each value of α separately, we easily realize that, contrary to the sharp interface case, the minimizer of Eq. (5.2) in the space of functions with a fixed number of jumps *is not independent of* α, since the shape of the transition profile from $u=-1$ to $+1$ depends in general on α. It would be interesting to understand whether this is just a technical issue or whether the difficulty arises from the fact that positive temperatures states are in general not simple periodic. We hope to come back to this issue in a future publication.

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APPENDIX: NONDEGENERACY OF THE MINIMIZERS

In this appendix we show that, for any $\alpha > 0$, the minimizers w of $E_\alpha(u) = \int_0^L dx u(x) W_{\alpha,u}(x)$ on the subspace of functions with exactly M jumps are associated to a nondegenerate sequence of jump points, $z_0 \equiv 0 < z_1 < \cdots < z_M < L \equiv z_{M+1}$; in other words, $z_j = z_{j+1}$ does not occur. Moreover, $W_{\alpha,w}(z_i)$, $i = 1, \ldots, M$, is independent of i, as claimed in Sec. III, right after Eq. (3.4).

Given any u with exactly M jumps (not necessarily a minimizer), let us identify it with its (possibly degenerate) sequence of jumps. This space of ordered sequences is clearly compact, so we have at least one minimizing sequence, which can, in principle, be degenerate; let us denote it by $0 \le z_1 \le \cdots \le z_M \le L$. If this sequence is degenerate, let $0 < \tilde{z}_1 < \cdots < \tilde{z}_{M_0} < L$, $M_0 < M$, be the nondegenerate ordered subsequence of $(z_1, \ldots, z_M)$. That is, we throw away the degenerate jumps at $z_j = z_{j+1}$. In this case, let us denote by ϕ the nondegenerate function belonging to the subspace of functions with M_0 jump points, associated to the sequence $\tilde{z}_1, \ldots, \tilde{z}_{M_0}$. Clearly, $\int_0^L dx w(x) W_{\alpha,w}(x) = \int_0^L dx \phi(x) W_{\alpha,\phi}(x)$ and ϕ is a minimizer of E_α in the subspace of functions with M_0 jumps. With some abuse of notation, we shall denote the energy of this nondegenerate configuration, as a function of the position of its jump points, by $E_\alpha(\tilde{z}_1, \ldots, \tilde{z}_M)$. By minimality, $\partial_\varepsilon E_\alpha(\tilde{z}_1, \ldots, \tilde{z}_i + \varepsilon,\ \tilde{z}_{i+1} + \varepsilon, \ldots, \tilde{z}_M)|_{\varepsilon=0} = 0$, which implies that $W_{\alpha,\phi}(z_i)$ is independent of i, with $i = 1, \ldots, M_0$.

Now, the potential $W_{\alpha,\phi}$ is concave in the intervals where ϕ is positive, and convex in the intervals where ϕ is negative, as we shall now prove. Assume that $\tilde{z}_i < x < \tilde{z}_{i+1}$ is such that $\phi(x) = +1$; in this case, rewriting $\phi(x) = -1 + 2\chi_\phi(x)$, with χ_ϕ the characteristic function of the region where $\tilde{\phi}$ is positive, we have that $W_{\alpha,\phi}(x) = -2\alpha^{-1} + 2\int_{\mathbb{R}} dy e^{-\alpha|x-y|} \chi_\phi(x)$, from which it is apparent that $W_{\alpha,\phi}(x)$ is convex, being the superposition of strictly convex functions. A similar proof applies to the case where x is such that $\phi(x) = -1$. As a consequence, there is exactly one strict internal maximum of the potential in every interval where the minimizer is positive, and exactly one strict internal minimum in every interval where the minimizer is negative. Therefore, we can always decrease the total potential energy by adding $M - M_0$ nondegenerate jumps, sufficiently close to each other and sufficiently close to, say, the left boundary of the big box $[0, L]$; this contradicts the assumption that w is a minimizer in the subspace of configurations with M jumps, and proves the claim.

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